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Subcritical convective instability

Part 2. Spherical shells

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In this paper we consider the effect of internal heat generation and a spatial variation of the gravity field on the onset of thermal convection in spherical shells. If the temperature gradient and gravity fields have the same spatial variation, then initially quiet fluids are subcritically stable. For these flows the effect of inertially non-linear disturbances is not destabilizing if the Rayleigh number is below the critical value set by linear theory plus 'exchange of stabilities'. For subcritically-stable flows a principle of exchange of stabilities is not necessary; a stronger statement of stability for the same stability limit can be made. For the many cases calculated here in which subcritical instabilities can exist, the difference between the linear and energy limits is small and can be contracted only toward the energy limit by an improved linear theory.

1. Introduction

In this application of energy theory we consider convective instability in spherical shells. Subcritical instabilities do not exist when the gravity and temperature-gradient fields have the same variation. Here, energy theory coincides with linear theory plus 'exchange of stabilities'. In these cases the energy theory is necessarily superior, physically, in that it guarantees stability to inertially non-linear disturbances, and mathematically, in that the problems generated are as simple as in the linear theory but do not involve assumptions of questionable validity. The class of basic states considered are chosen for easy comparison with linear theory. Thus, we have for the most part considered cases which are available in Chandrasekhar's (1961) monograph. Chandrasekhar's linear results have been recalculated numerically,† extended to some cases not treated by him, and compared to corresponding results calculated from energy theory. Possible ranges for subcritical instabilities are found only when the gravity and temperature-gradient distribution have a different variation. As in the internally heated fluid layer considered in part 1, the ranges of possible subcritical instabilities, when they exist at all, are confined to a very narrow band of Rayleigh numbers.

† Chandrasekhar's stability results are in excellent agreement (1 %) with the values calculated by numerical methods. These latter are used for our comparison of the energy and linear theory. Extended results have been tabulated by Carmi (1966).

2. Convective instability in spherical shells—perturbation and energy

We assume a spherically symmetric and radial gravitational field,

$$\mathbf{g} = -G(r^*)\,\mathbf{r}^*, \tag{1}$$

and temperature field,

$$T = \sigma_0 - \sigma_2 r^{*2} + \sigma_1/r^*, \tag{2}$$

where the temperature field is determined by a combination of an imposed temperature difference at the boundaries and a distribution of heat sources satisfying

$$\kappa\nabla^2 T = -\epsilon. \tag{3}$$

The temperature gradient is given by

$$\nabla T = \frac{\mathbf{r}^*}{r^*}\frac{\partial T}{\partial r^*} = -2\mathbf{r}^*(\sigma_2 + \tfrac{1}{2}\sigma_1/r^{*3}) = -2\mathbf{r}^* B(r^*), \tag{4}$$

where $\sigma_2 = \frac{1}{6}\epsilon/\kappa$. With these assumptions, the system of partial differential equations governing small perturbations (Chandrasekhar 1961, p. 221) is

$$0 = -\nabla p^* + \alpha G(r^*)\,\mathbf{r}^*\theta + \nu\nabla^2\mathbf{u}, \tag{5}$$

$$0 = 2B(r^*)\,\mathbf{u}\,.\,\mathbf{r}^* + \kappa\nabla^2\theta, \tag{6}$$

where partial-time derivatives have been set to zero in accord with the assumption that the principle of exchange of stabilities prevails. It is convenient to introduce the variables

$$\left.\begin{aligned} r = r^*/r_1^*, \quad G_1 = G(r_1^*), \quad B_1 = B(r_1^*), \quad c(r) = G(r^*)/G_1, \\ b(r) = B(r^*)/B_1, \quad \mathbf{v} = \mathbf{u}(2B_1\nu/\alpha G_1\kappa)^{\frac{1}{2}}, \quad R_a^+ = 2\alpha B_1 G_1 r_1^{*6}/\nu\kappa. \end{aligned}\right\} \tag{7}$$

With these substitutions (5) and (6) become

$$0 = -\nabla p + (Ra^+)^{\frac{1}{2}}\, c(r)\,\mathbf{r}\theta + \nabla^2\mathbf{v}, \tag{8}$$

$$0 = (Ra^+)\, b(r)\,\mathbf{r}\,.\,\mathbf{v} + \nabla^2\theta, \tag{9}$$

and these are to be solved subject to (3) or (4), (5), and (10) of part 1.

We wish to compare (8) and (9) with the Euler–Lagrange equations (21) and (22) of part 1. To facilitate this comparison we leave g and β unspecified. Then

$$\lambda\nabla\psi + \mathbf{f} = \lambda\nabla T/\beta + \mathbf{g}/g = -r_1^*\,\mathbf{r}\{\lambda b(r)\, 2B_1/\beta + c(r)\, G_1/g\},$$

and with $\beta = 2B_1$ and $g = G_1$,

$$\lambda\nabla\psi + \mathbf{f} = -r_1^*\,\mathbf{r}\{\lambda b(r) + c(r)\}. \tag{10}$$

Now we replace $r_1^* R_\lambda$ with R_λ and rewrite (21) and (22) of part 1 as

$$0 = -\nabla p + \tfrac{1}{2}R_\lambda\{\lambda b(r) + c(r)\}\,\mathbf{r}\theta + \nabla^2\mathbf{v}, \tag{11}$$

and

$$0 = \tfrac{1}{2}(R_\lambda/\lambda)\,(\lambda b(r) + c(r))\,\mathbf{r}\,.\,\mathbf{v} + \nabla^2\theta. \tag{12}$$

These equations are also to be solved subject to (3) or (4), (5), and (10) of part 1.

3. Subcritical instabilities—exchange of stabilities

It is easy to verify that with the form of the gravitational and temperature field given in § 2 and with $g = G_1$ and $\beta = 2B_1$, the λ of (26) in part 1 is given by

$$\lambda = \frac{\int_{\mathscr{V}'} c(r)\,\mathbf{r}\,.\,\mathbf{v}\theta}{\int_{\mathscr{V}'} b(r)\,\mathbf{r}\,.\,\mathbf{v}\theta}. \tag{13}$$

With $b(r) = c(r)$, $\lambda = 1$, and the systems of partial differential equations governing the energy (equations 11 and 12) are identical to those governing small perturbations (equations 8 and 9) when partial time derivatives are set to zero. It follows that subcritical instabilities are not possible when $b(r) = c(r)$. $\widetilde{Ra}^+$ is then the critical Rayleigh number below which the flow is definitely stable, and above which the flow is definitely unstable.

As is known (Joseph 1966), rigid rotation cannot decrease $\widetilde{Ra}^+$ and hence, cannot destabilize the flow if $Ra^+ < \widetilde{Ra}^+$.

A similar remark applies to the possibility that instability may occur as overstability with $Ra^+ < \widetilde{Ra}^+$. This is clearly not possible when $b(r) = c(r)$. We note that the principle of exchange of stabilities has been proved (see Chandrasekhar 1961) only for the special case $b = c = 1$. It follows that when $b(r) = c(r)$, the lowest critical Rayleigh number can be obtained from the time-independent perturbation equations which in this case coincide with the energy equations. If overstability can occur, it must occur for $Ra^+ > \widetilde{Ra}^+$.

It is perhaps of even greater interest that for the large number of cases considered by Chandrasekhar in which $b(r) = c(r)$, our calculation reveals only a very small range of Rayleigh numbers as possible candidates for subcritical instabilities. Hence, if in these cases it is possible for overstability to set in at values of the Rayleigh number less than those given by the time-independent perturbation result, these values must be confined to the narrow band in which subcritical instabilities are possible.

4. Eigenvalue problem for spherical shells

We follow Chandrasekhar in reducing the partial differential equations (8) and (9) to a coupled set of ordinary differential equations. First, take the curl of (8) twice; then, scalar multiply the result of the two curl operations with the position vector. We write the result in spherical polar co-ordinates (r, A, B),

$$0 = -(Ra^+)^{\frac{1}{2}}\,c(r)\,L^2\theta + \nabla^4\mathbf{v}\,.\,\mathbf{r}, \tag{14}$$

where $$L^2 = r^2\left\{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \nabla^2\right\} = -\frac{1}{\sin A}\frac{\partial}{\partial A}\sin A\frac{\partial}{\partial A} - \frac{1}{\sin^2 A}\frac{\partial^2}{\partial B^2}.$$

The dependent variables θ and $\mathbf{v}\,.\,\mathbf{r}$ are next expanded in a complete set of spherical harmonics

$$\left.\begin{aligned} \mathbf{v}\,.\,\mathbf{r} &= W(r)\,Y_l^m(A,B), \\ \theta &= \Theta\,Y_l^m(A,B), \end{aligned}\right\} \tag{15}$$

where (see Chandrasekhar, p. 223)

$$Y_l^m(A,B) = P_l^m(\cos A)\, e^{\pm imB}, \tag{16}$$

and P_l^m are the associated Legendre polynomials.

Substitution of (15) into (14) and (9) yields

$$\mathscr{D}_l^2 W = (Ra^+)^{\frac{1}{2}} c(r)\, l(l+1)\,\Theta, \tag{17}$$

and

$$\mathscr{D}_l \Theta = -(Ra^+)^{\frac{1}{2}} b(r)\, W, \tag{18}$$

where

$$\mathscr{D}_l = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}. \tag{19}$$

An identical procedure may be followed to obtain the equations.

$$\mathscr{D}_l^2 W = \tfrac{1}{2} R_\lambda \{\lambda b(r) + c(r)\}\, l(l+1)\,\Theta, \tag{20}$$

$$\mathscr{D}_l \Theta = -\tfrac{1}{2}(R_\lambda/\lambda)\,(\lambda b(r) + c(r))\, W, \tag{21}$$

from (11) and (12). Finally with

$$F = (Ra^+)^{\frac{1}{2}}\, l(l+1)\,\Theta, \quad Ra^+ = C_l, \tag{22}$$

in (17) and (18) and

$$F = R_\lambda\, l(l+1)\,\Theta\lambda^{\frac{1}{2}}, \quad R_\lambda^2 = C_{l\lambda}, \tag{23}$$

in (20) and (21) we obtain the perturbation equations

$$\mathscr{D}_l^2 W = c(r) F, \tag{24}$$

$$\mathscr{D}_l F = -l(l+1)\, C_l\, b(r)\, W, \tag{25}$$

and the energy equations

$$\mathscr{D}_l^2 W = \{\lambda b(r) + c(r)\} F/2\lambda^{\frac{1}{2}}, \tag{26}$$

$$\mathscr{D}_l F = -l(l+1)\, C_{l\lambda}\{\lambda b(r) + c(r)\}\, W/2\lambda^{\frac{1}{2}}. \tag{27}$$

At the boundaries ($r = 1, \eta$), the functions F and W satisfy

$$W = F = 0, \tag{28}$$

and

$$W' = dW/dr \quad \text{on rigid surfaces,} \tag{29}$$

$$W'' = d^2W/dr^2 \quad \text{on free surfaces.} \tag{30}$$

A full discussion of these boundary conditions is given by Chandrasekhar (pp. 224–225).

5. Solutions

We solve the eigenvalue problems (24), (25) and (26), (27) plus boundary conditions numerically as an initial-value problem. The solutions are represented by the linear combinations

$$\begin{Bmatrix} W \\ F \end{Bmatrix} = \sum_{i=1}^{3} A_i \begin{Bmatrix} W_i \\ F_i \end{Bmatrix}. \tag{31}$$

At the inner radius we require W_i and F_i to satisfy separately (28), (29) or (30). In addition, three arbitrary conditions at $r = \eta$ are set as follows:

$$\left.\begin{array}{llll} F_1' = 0, & F_2' = 0, & F_3' = 1, & \\ W_1''' = 0, & W_2''' = 1, & W_3''' = 0, & \\ \left.\begin{array}{l} W_1'' \\ W_1' \end{array}\right\} = 1, & \left.\begin{array}{l} W_2'' \\ W_2' \end{array}\right\} = 0, & \left.\begin{array}{l} W_3'' \\ W_3' \end{array}\right\} = 0 & \begin{array}{l} \text{(rigid surface,} \\ \text{(free surface),} \end{array} \end{array}\right\} \tag{32}$$

where the primes indicate differentiation with respect to r.

A linear combination of the solutions (31) will satisfy the boundary conditions at a rigid (say) outer ($r = 1$) surface if

$$\Delta(l, C_l, \eta) = \begin{vmatrix} F_1 & F_2 & F_3 \\ W_1 & W_2 & W_3 \\ W_1' & W_2' & W_3' \end{vmatrix} = 0,$$

for the perturbation problem (24) and (25), and

$$\Delta(l, C_{l\lambda}, \eta, \lambda) = \begin{vmatrix} F_1 & F_2 & F_3 \\ W_1 & W_2 & W_3 \\ W_1' & W_2' & W_3' \end{vmatrix} = 0,$$

for the energy problem (26) and (27). We are interested primarily in the smallest values C_l and $C_{l\lambda}$ for fixed λ and η and must search Δ for the minimum over l.

The elements of Δ are provided numerically by forward integration using the Runge–Kutta–Gill method (Harris & Reid 1964, Sparrow 1964).

The flow is, as we have explained, stable if

$$Ra^+ < \min_{l=1,2,\ldots} C_{l\lambda}$$

for any given $\lambda > 0$. We next seek the best λ as that for

$$\max_{\lambda>0} \min_{l} C_{l\lambda}. \tag{33}$$

This is given by (13) as

$$\lambda = \frac{\int_\eta^1 c(r)\,\mathbf{r}\,.\,\tilde{\mathbf{v}}\tilde{\theta}}{\int_\eta^1 b(r)\,\mathbf{r}\,.\,\tilde{\mathbf{v}}\tilde{\theta}},$$

which may be crudely estimated by the first mean-value theorem of integral calculus. Assuming that $\mathbf{r}\,.\,\tilde{\mathbf{v}}\tilde{\theta}$ is one signed on $(\eta, 1)$, we find that

$$\lambda_{\text{est}} = c(\bar{r})/b(\bar{\bar{r}}), \tag{34}$$

where $\bar{r}$ and $\bar{\bar{r}}$ are mean values. This estimate we consistently formed with the guess that both mean values may be replaced with an arithmetic mean,

$$\bar{r} = \bar{\bar{r}} = \tfrac{1}{2}(1+\eta). \tag{35}$$

We next search the neighbourhood of λ_{est} for that λ which gives $\min_l C_{l\lambda}$ a maximum. In view of the crudity of the estimate, the accuracy with which it gives the correct λ is remarkable (see tables 1 to 4).

6. Results

Some representative results are summarized in tables 1–4 and in figures 1 and 2. We have reported on cases for which the variation of $b(r)$ and $c(r)$ may be compared directly with generally available results of Chandrasekhar. A greater variety of results and the computer program used in obtaining these results are to be found in Carmi (1966). A brief summary of the meaning of the considered

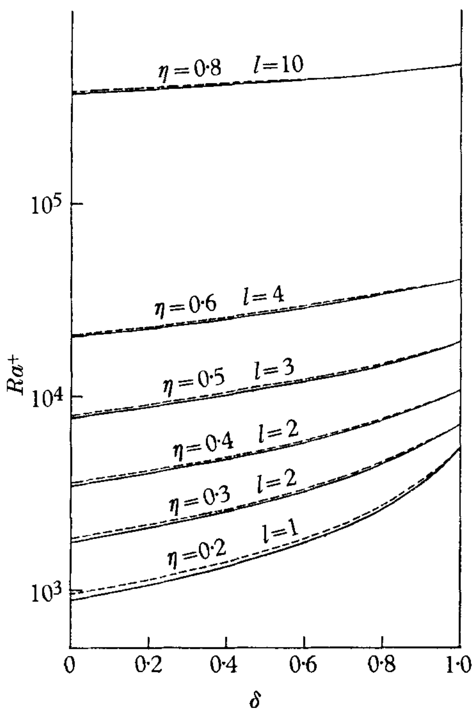

FIGURE 1

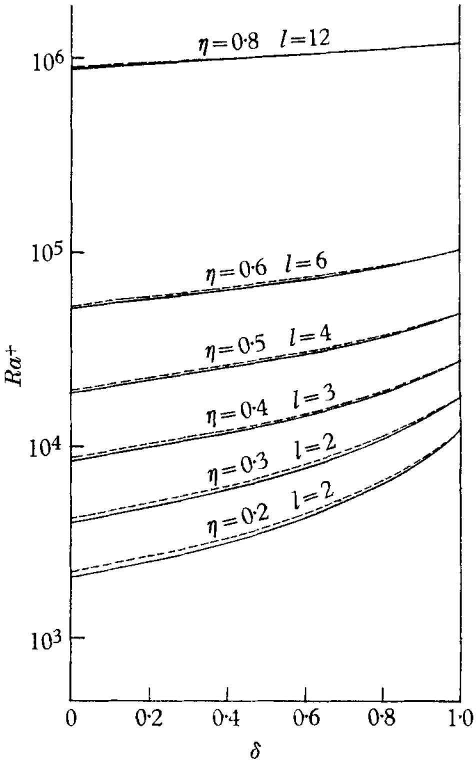

FIGURE 2

FIGURE 1. Variation of the Rayleigh number as a function of δ for the linear and energy theories. Free-free surfaces. ---, linear theory; ——, energy theory.

FIGURE 2. Variation of the Rayleigh number as a function of δ for the linear and energy theories. Rigid-rigid surfaces. ---, linear theory; ——, energy theory.

Radius ratio	Mini-mizing harmonic	λ_{est} equation (34)	Best λ	$Ra^+ \times 10^{-3}$ Energy theory	$Ra^+ \times 10^{-3}$ Linear theory
Free-free surfaces (cf. Chandrasekhar 1961, p. 250)					
0·2	1	1·67	1·671	3·132	3·149
0·3	2	1·54	1·520	4·690	4·708
0·4	2	1·43	1·426	7·645	7·662
0·5	3	1·33	1·330	14·46	14·48
0·6	4	1·25	1·256	32·63	32·66
0·8	10	1·11	1·110	459·5	459·6
Rigid-rigid surfaces					
0·2	2	1·67	1·690	7·525	7·568
0·3	2	1·54	1·420	11·67	11·71
0·4	3	1·43	1·420	19·35	19·39
0·5	4	1·33	1·345	37·07	37·12
0·6	6	1·25	1·250	84·37	84·44
0·8	12	1·11	1·110	1129	1129

TABLE 1. Critical Rayleigh numbers for non-linear gravity distribution ($c = 1/r$) and linear temperature gradient ($b = 1$)

gravity and heat-source variations follows. The implication of these variations in astrophysical and geophysical applications may be found in Chandrasekhar and in references cited therein.

First we recall that

$$\mathbf{g} = -G_1 r_1^* \mathbf{r} c(r),$$

and

$$\nabla T = -2r_1^* B_1 \mathbf{r} b(r),$$

Radius ratio	Minimizing harmonic	λ_{est} equation (34)	Best λ	$Ra^+ \times 10^{-3}$ Energy theory	$Ra^+ \times 10^{-3}$ Linear theory
Free-free surfaces (cf. Chandrasekhar 1961, p. 250)					
0·2	1	1·52	1·708	3·107	3·157
0·3	2	1·38	1·440	4·999	5·033
0·4	2	1·27	1·309	8·357	8·379
0·5	3	1·20	1·210	15·91	15·93
0·6	4	1·14	1·140	35·68	35·69
0·8	10	1·05	1·040	484·3	484·3
Rigid-rigid surfaces					
0·2	2	1·52	1·710	7·586	7·707
0·3	2	1·38	1·480	12·49	12·57
0·4	3	1·27	1·310	21·26	21·31
0·5	4	1·20	1·220	40·93	40·98
0·6	6	1·14	1·140	92·42	92·46
0·8	12	1·05	1·05	1191	1191

TABLE 2. Critical Rayleigh numbers for non-linear gravity distribution ($c = (6+1/r^3)/7$) and linear temperature gradient ($b = 1$)

Radius ratio	Minimizing harmonic	λ_{est} equation (34)	Best λ	$Ra^+ \times 10^{-3}$ Energy theory	$Ra^+ \times 10^{-3}$ Linear theory
Free-free surfaces (cf. Chandrasekhar, 1961, p. 250)					
0·2	1	4·63	6·50	0·8732	0·9314
0·3	2	3·65	4·34	1·737	1·813
0·4	2	2·92	3·28	3·419	3·497
0·5	3	2·37	2·53	7·723	7·828
0·6	4	1·95	2·03	20·29	20·44
0·8	10	1·36	1·38	370·1	370·6
Rigid-rigid surfaces					
0·2	2	4·63	6·57	2·125	2·270
0·3	2	3·65	4·60	4·174	4·320
0·4	3	2·92	3·34	8·605	8·786
0·5	4	2·37	2·55	19·69	19·93
0·6	6	1·95	2·00	52·38	52·74
0·8	12	1·36	1·38	908·4	909·8

TABLE 3. Critical Rayleigh numbers for non-linear gravity distribution ($c = 1/r^3$) and linear temperature gradient ($b = 1$)

so that $b = c = 1$ implies a linear-gravitational and temperature-gradient field directed radially inward. Non-constant b or c implies that one (or both) of the fields has a non-linear radial variation. The form of b is further restricted by the uniform heat generation. In general,

$$b(r) = B(r^*)/B(r_1^*) = (\sigma_2 + \tfrac{1}{2}\sigma_1/r^{*3})/B_1 = \delta + (1-\delta)/r^3, \tag{36}$$

where $\delta = \sigma_2/B_1$ is a heat-source parameter. When $\delta = 0$ there are no heat sources, and the temperature of the quiescent motion is maintained externally. When

Heat source parameter (δ)	λ_{est}	Best λ	$Ra^+ \times 10^{-3}$ Energy theory	$Ra^+ \times 10^{-3}$ Linear theory
Free-free surfaces, minimizing harmonic = 3				
0·0	0·42	0·396	7·723	7·828
0·2	0·47	0·450	8·786	8·885
0·4	0·55	0·525	10·18	10·27
0·6	0·64	0·625	12·10	12·16
0·8	0·78	0·770	14·88	14·91
1·0	1	1	19·22	19·22
Rigid-rigid surfaces, minimizing harmonic = 4				
0·0	0·42	0·390	19·69	19·93
0·2	0·47	0·445	22·43	22·65
0·4	0·55	0·520	26·04	26·23
0·6	0·64	0·620	31·00	31·15
0·8	0·78	0·770	38·23	38·30
1·0	1	1	49·62	49·62

TABLE 4. Critical Rayleigh numbers for linear gravity distribution ($c = 1$) and non-linear temperature gradient with variable heat source intensity

$(b = \delta + (1-\delta)/r^3)$ for $\eta = 0{\cdot}5$ and $\lambda_{est} = \{\delta + (1-\delta)/0{\cdot}75^3\}^{-1}$

$\delta = 1$ there is a uniform temperature common to the inner-outer boundary, and the temperature variation is maintained solely by the uniform heat-source distribution.

The cases considered here are conveniently grouped as follows:

(i) $b(r) = c(r)$. Here $\lambda = 1$ and the linear and energy results coincide. Subcritical instabilities are not possible.

(ii) The temperature distribution is linear, $b(r) = 1$, but the gravity distribution is non-linear: $c(r)$ varies. The results are summarized in tables 1–3. Subcritical instabilities are possible only in a narrow Rayleigh-number band which grows wider as the gravity variation becomes more intense.

(iii) The temperature distribution is non-linear, $b(r)$ varies, but the gravity distribution is linear; $c(r) = 1$. The results are summarized in table 4 and in figures 1 and 2. Subcritical instabilities are possible only in a narrow Rayleigh number band which grows narrower with the magnitude of the heat-source intensity.

7. Discussion

A real value of the energy method is its ability to delineate possible regions of subcritical instability. For those flows in which these instabilities are not possible, the effect of inertially non-linear disturbances is not destabilizing if the Rayleigh number is below the critical value set by linear theory plus 'exchange of stabilities'. For subcritically stable flows, then, a principle of exchange of stabilities is not necessary; a stronger statement of stability for the same stability limit can be made. Roughly speaking, if the temperature gradient and gravity fields have the same spatial variation, then initially quiet fluids are subcritically stable.

For those flows which are not subcritically stable, instabilities may be produced by inertially non-linear disturbances under circumstances which would require small-amplitude disturbances to decay. The Rayleigh-number ranges for which such a situation is possible is in the purview of the energy theory provided that the true linear result is known. It is usual, under the conditions that prevail when subcritical instabilities can exist, that exchange of stability cannot be proved but is assumed for the calculation of the linear limit. It may be that the linear limit calculated in this way could give greater critical Rayleigh numbers than the linear limit associated with the unsteady linear-perturbation equations. It follows, that unless we have a guarantee of the validity of the linear limit, the range of possible subcritical instabilities is uncertain at the linear end. It is comforting that in many of these cases, this band of possible subcritical instabilities is already narrow and could only be contracted toward the energy limit by a better linear theory.

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